Scattering of scalar tardyons and tachyons from a Schwarzschild black hole

1982 J. Phys. A: Math. Gen. 15 3181

(http://iopscience.iop.org/0305-4470/15/10/022)

J. Phys. A: Math. Gen. **15** (1982) 3181–3185. Printed in Great Britain

Scattering of scalar tardyons and tachyons from a Schwarzschild black hole

S Banerji and D R Mandal

Department of Physics, The University of Burdwan, Burdwan 713104, India

Received 16 November 1981

Abstract. We have set up the Klein–Gordon equation in the background of the Schwarzschild curved space–time and studied the scattering of radial tardyons and tachyons from a black hole. We also show that black holes of mass below 7×10^{14} g may contain bound states of tardyons of pion mass which will be unstable on account of the presence of an attractive r^{-4} term.

1. Introduction

Dhurandhar (1978) investigated the behaviour of tachyonic scalar waves near a Schwarzschild black hole. We shall here consider spinless tardyons and tachyons obeying the Klein–Gordon equation, which will be treated as an equation for the probability amplitude and hence the problems associated with the negative energy solutions and attendant features of vacuum polarisation effects and pair creation will be ignored. Unruh (1976) considered scattering of particles from small black holes with dimensions much less than the particle wavelength. However, here we do not make this assumption.

Writing the Klein–Gordon equation in a Schwarzschild background, separating the variables in $\psi(r,\theta,t)$ in the usual way and introducing dimensionless units $r = RM_0c/\hbar$, $m = GMM_0/c\hbar$, $\omega = \Omega\hbar/M_0c^2$, ($M$ and M_0 being the mass parameters of the black hole and the scalar particle respectively) we obtain (for details see Dhurandhar 1978):

$$\frac{d^2\psi_l^\Omega}{dr^2}+\frac{2m}{r(r-2m)}\frac{d\psi_l^\Omega}{dr}+\frac{r^2}{(r-2m)^2}\times\left[\omega^2\mp\left(1-\frac{2m}{r}\right)-\frac{l(l+1)}{r^2}\left(1-\frac{2m}{r}\right)-\left(1-\frac{2m}{r}\right)\frac{2m}{r^3}\right]\psi_l^\Omega=0. \tag{1}$$

The first sign refers to a tardyon and the second to a tachyon, which has rest mass i M_0.

We now use the transformation given by Talukdar *et al* (1981):

$$\psi_l^\Omega(r)=[r/(r-2m)]^{1/2}\varphi_l^\Omega(r) \tag{2}$$

to obtain

$$\frac{d^2\varphi_l^\Omega}{dr^2}+\frac{r^2}{(r-2m)^2}\left[\omega^2\mp\left(1-\frac{2m}{r}\right)+\frac{m^2}{r^4}-\frac{l(l+1)}{r^2}\left(1-\frac{2m}{r}\right)\right]\varphi_l^\Omega=0. \tag{3}$$

0305-4470/82/103181+05$02.00

Thus we can eliminate the term involving the first derivative without making any assumption regarding the relative magnitudes of M and M_0 unlike Dhurandhar.

2. Scattering of particles from a black hole

Let us consider radial particles, $l = 0$; then (3) reduces to (omitting sub and superscripts of φ):

$$\frac{d^2\varphi}{dr^2}+\frac{r^2}{(r-2m)^2}\{\omega^2 - V(r)\}\varphi = 0, \tag{4}$$

where

$$V(r) \equiv \pm(1-2m/r) - m^2/r^4. \tag{4a}$$

$V(r)$ may be regarded as the effective potential due to the curvature of space–time. For tardyons it is negative everywhere within the black hole. In a small region surrounding $r = 0$ the last term m^2/r^4 predominates. For tachyons on the other hand the potential rises to a maximum positive value $V_m = m^{2/3}(2^{2/3} - 2^{-4/3}) - 1$ and then falls to $-\infty$ at $r = 0$. The fraction, a, of the black hole radius where the $1/r^4$ term predominates over the $1/r$ term is $(4m)^{-2/3}$. If we take for example $M = M_\odot$ and $M_0 = 2.4 \times 10^{-25}$ g then $a \sim 10^{-12}$. Further, the height of the potential maximum increases with M. Therefore, we shall neglect the last term in (4a) so that

$$\frac{d^2\varphi}{dr^2}+\frac{r^2}{(r-2m)^2}\left(k^2 \pm \frac{2m}{r}\right)\varphi = 0, \tag{5}$$

where $k^2 = \omega^2 \mp 1$. Putting $x = 2m - r$, we obtain

$$\frac{d^2\varphi}{dx^2}+\left(k^2 - \frac{2m\alpha}{x} - \frac{\mu^2 - \frac{1}{4}}{x^2}\right)\varphi = 0, \tag{6}$$

where $\alpha = 2\omega^2 \mp 1$ and $\mu^2 = \frac{1}{4}(1 - 16m^2\omega^2)$; accordingly μ is real (or pure imaginary) for $\omega < \frac{1}{4}m$ (or $\omega > \frac{1}{4}m$). Using the transformation $\phi(x) = x^{\mu+1/2}\exp(\mathrm{i}kx)f(x)$ the equation for f becomes the confluent hypergeometric equation as shown by Dhurandhar (1978). However, Dhurandhar did not use the exact solution to investigate his problem. He used the WKB approximation instead, whose validity is not assured for all masses of the black hole.

The exact solution for the radial part of the wavefunction can be written as

$$\begin{aligned}\psi(x) = {} & C_1(2m-x)^{1/2}x^{\mu}\exp(\mathrm{i}kx)F(\mu+\tfrac{1}{2}+\mathrm{i}\alpha m/k; 2\mu+1; -2\mathrm{i}kx) \\ & + C_2(2m-x)^{1/2}x^{-\mu}\exp(\mathrm{i}kx)F(-\mu+\tfrac{1}{2}+\mathrm{i}\alpha m/k; -2\mu+1; -2\mathrm{i}kx).\end{aligned} \tag{7}$$

For real $\mu < 0$ (it suffices to consider $\mu < 0$ in view of the symmetry under $\mu \to -\mu$) the first term becomes infinite at $x = 0$ (the event horizon) and hence the good behaviour of $\psi\,(x)$ demands $C_1 = 0$ (and similarly $C_2 = 0$ for $\mu > 0$). It may be noted that the deletion of the $1/r^4$ term has led to an equation for which $r = 0$ (i.e. $x = 2m$) is in fact an ordinary point and hence the behaviour in the neighbourhood of $r = 0$ imposes no further restriction on the solution. For $\mu = 0$ and for pure imaginary values of the boundary condition, it is interesting to observe that no restriction is imposed on the wavefunction and henceforth attention will be confined to real values of μ.

2.1. *Tardyons*

When μ is real ($\omega < \frac{1}{4}m$) we have for scattering states (k real, $\omega > 1$), $M_0c^2 < E < 6\times 10^{16}/M$ MeV. If for the sake of illustration we take $M_0c^2 \sim 134$ MeV corresponding to the pion rest mass this implies $M < 4.5\times 10^{14}$ g.

Taking the asymptotic form of the S-wave solution ψ at $r\to\infty$, $x\to-\infty$ we obtain

$$\psi(x)\to\left(\frac{2m-x}{x}\right)^{1/2}\left[\exp(\mathrm{i}kx)\frac{\Gamma(2\mu+1)}{\Gamma(\mu+\frac{1}{2}-\mathrm{i}\alpha m/k)}x^{-\mathrm{i}\alpha m/k}\right.$$
$$\left.+\exp(-\mathrm{i}kx)\frac{\Gamma(2\mu+1)}{\Gamma(\mu+\frac{1}{2}+\mathrm{i}\alpha m/k)}x^{\mathrm{i}\alpha m/k}\right]. \tag{8}$$

The first term gives the incoming wave and the second gives the outgoing wave. We find that the plane wave is distorted even at large distances from the centre. This also happens in the case of Coulomb (Rutherford) scattering and is due to the presence of a $1/r$ term in both cases.

Now the absolute value of the scattering matrix element, S, is given by

$$S=\frac{|\text{amplitude of the outgoing wave}|}{|\text{amplitude of the incoming wave}|}=1.$$

We may conclude therefore that 'mini' black holes of mass $<4.5\times 10^{14}$ g scatter the low-energy tardyons completely without absorption. This is contrary to the classical result that all tardyons incident on a black hole are absorbed. However, in quantum mechanics scattering is present even when we take a perfectly absorbing body (see e.g. Blatt and Weisskopf 1952, p 324).

While this result would at first appear to be simply a consequence of the reality of the potential leading to the requirement of a unitary scattering operator, in general, however, even for a real potential with singular and pathological behaviour this text book result is not valid (Pearson 1975). In the present case care must be exercised in view of the singularity at the origin which makes the Hamiltonian pseudo-Hermitian (Landau and Lifshitz 1965, Morse and Feshbach 1953) corresponding to the classical 'winding into the centre' problem. In fact the problem arising for an attractive $1/r$ potential in the relativistic case is analogous (in so far as such pathologies are concerned) to the $1/r^2$ attractive potential for the non-relativistic Schrödinger equation.

The limitation on the mass is reminiscent (and owes its origin to the same source) of the result in relativistic Coulomb scattering in quantum mechanics wherein for $Z>(e^2/\hbar c)^{-1}=137$ (superheavy elements) pathologies develop (Berestatskii *et al* 1971 and Bjorken and Drell 1964).

2.2. *Tachyons*

In this case k is always real but μ is real for $\omega<\frac{1}{4}m$ and $S=1$ in this case as before. Unlike the above case, there is no upper limit to the mass of black holes which scatter without absorption all particles with energy below a certain limit. We should, however, remember that the energy of a tachyon decreases as the velocity increases, reducing to zero for $v\to\infty$.

Raychaudhuri (1974) found classically that all radial tachyons penetrate the Schwarzschild black hole and are then reflected from a distance $R=2M/k^2$ where k represents the momentum per unit mass parameter of the tachyon.

3. Energy levels of tardyons bound by the black hole

For tardyons, k becomes imaginary when $\omega < 1$. This can occur only in an attractive field. If we put $k = iK$, then $\exp(ikx) = \exp(-Kx)$ and $\exp(-ikx) = \exp(Kx)$. Hence in (8) the first term goes to infinity as $r \to \infty$ ($x \to -\infty$) while the second term decays with distance, so the wavefunction will be admissible only when the coefficient of the first term vanishes, i.e. when $\mu + \frac{1}{2} - \alpha m/K = -n$, where $n = 0, 1, 2$ (an integer) i.e.

$$n + \tfrac{1}{2}[1 + (1 - 16m^2\omega^2)^{1/2}] = m(2\omega^2 - 1)/(1 - \omega^2)^{1/2}. \tag{9}$$

We must have $\omega < \frac{1}{4}m$ in order that the second term on the left may be real because all other terms in (9) are real. Further, since the left-hand side is positive the right-hand term should also be positive, i.e. $\omega > 1/\sqrt{2}$. Hence for bound states to occur $m < 1/2\sqrt{2}$. For pions this implies $M < 7 \times 10^{14}$ g.

According to the classical theory, however, such bound states cannot exist within the black hole as all such particles will fall to the centre but here there exists a non-zero probability of finding such particles inside the event horizon. Even outside the black hole the smallest stable orbit in the classical theory has $R = 6M$. It should be noted, however, that we have neglected an attractive term $O(1/r^4)$. This may make the bound states unstable as we find in the nonrelativistic case (Landau and Lifshitz 1965, Morse and Feshbach 1953).

4. Conclusions

We have found an exact solution of the problem of a radial particle obeying the Klein–Gordon equation in the curved space–time of Schwarzschild, after neglecting a term $O(1/r^4)$. We found that 'mini' black holes of mass below 4.5×10^{14} g scatter, without absorption, all tardyons of pion mass having energy below a certain limit, while all black holes do so in the case of tachyons. This result has been obtained for the S partial wave ($l = 0$) which is adequate for a substantial range of energies in view of the fact that the radius of such a black hole would be $\sim 10^{-14}$ cm. Further work is in progress for determining the role of higher partial waves. The classical result on the other hand is that a black hole absorbs all the incident tardyons but none of the incident tachyons. However, for particles of energy above the limit mentioned the boundary condition at the event horizon puts no restriction on the wavefunction and the problem lacks complete definition. A more complete analysis which does not neglect the $1/r^4$ term will give us the scattering cross-section for such higher energy values. We have also found that black holes of mass below 7×10^{14} g may form bound states of tardyons of pion mass but in the classical case all tardyons fall to the centre once they are inside the event horizon.

Acknowledgments

Our thanks are due to Professor B Datta Roy and Dr A K Bhattacharya for stimulating and valuable discussions on quantum mechanical problems. Our thanks are also due to the Indian Space Research Organisation for financial support.

References

Berestatskii V B, Lifshitz E M and Pitaevski I P 1971 *Relativistic Quantum Theory* (Oxford: Pergamon) p 111
Bjorken J D and Drell S D 1964 *Relativistic Quantum Mechanics* (New York: McGraw-Hill) p 55
Blatt J M and Weisskopf V F 1952 *Theoretical Nuclear Physics* (New York: Wiley)
Dhurandhar S V 1978 *J. Math. Phys.* **19** 561
Landau L D and Lifshitz E M 1965 *Quantum Mechanics–Nonrelativistic Theory* (Oxford: Pergamon) p 113
Morse P M and Feshbach H 1953 *Methods of Theoretical Physics Part II* (New York: McGraw-Hill) p 1665
Pearson D B 1975 *Commun. Math. Phys.* **40** 125
Raychaudhuri A K 1974 *J. Math. Phys.* **15** 856
Talukdar B, Sen M and Sen D 1981 *J. Math. Phys.* **22** 377
Unruh W G 1976 *Phys. Rev.* **14D** 3251